Rough Surface Contact

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ABSTRACT
This paper studies the contact of general rough curved surfaces having nearly identical geometries, assuming the contact at each differential area obeys the model proposed by Greenwood and Williamson. In order to account for the most general gross geometry, principles of differential geometry of surface are applied. This method while requires more rigorous mathematical manipulations, the fact that it preserves the original surface geometries thus makes the modeling procedure much more intuitive. For subsequent use, differential geometry of axis-symmetric surface is considered instead of general surface (although this “general case” can be done as well) in Chapter 3.1. The final formulas for contact area, load, and frictional torque are derived in Chapter 3.2.

1. INTRODUCTION
The need of understanding rough surfaces contact has long been recognized. One primary focuses of the early studies is to predict real contact area as it varies with load. Since a rough surface is known to include layers of micro-asperities, the real area of contact can be extremely small comparing to the apparent area observed by our eyes and is very difficult to measure. This problem has been addressed and resolved for the first time by Archard, Greenwood and Williamson using novel fractal and statistical models to mathematically describe the microscopic surface structure. Their works have been the basis for various subsequent studies on contact mechanics, describing the surface geometry (asperities distribution, geometry) and material behavior (elastic, plastic flow) (Yastrebov et al. 2014). Recently, a deterministic approach to model rough surface contact has grown rapidly with the advance of computational capability, providing further insights to the study of contact mechanics.

The fact that only nominally flat rough surfaces case is focused has limited the scope of this model. One reason for this shortage was given by Greenwood and Trip, as generally the curvatures difference of curved surfaces creates a cluster effect which makes asperities interaction becomes significant. Thus an intensive analysis similar to those performed in the nominally flat rough surfaces contact is not frequently performed in the case of rough curved surfaces contact. Rather, the latter in only loosely studied through the inspection of axial contact between two rough curved surfaces having constant curvatures, by replacing them with a nominally flat rough surface and a smooth curved surface having anequivalent-curvature (Johnson 1985). This method although gives a quick approximation of pressure distribution, it does not allow one to account for:

- More general analysis, such as the contact is non-axial or the surfaces have varying curvatures

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- More detailed analysis, such as the true distribution of contact pressure which is important to the calculation of cumulative frictional torque in rotating parts.

This paper studies the contact of general rough curved surfaces having nearly identical geometries, assuming that the contact at each differential area obeys the model proposed by Greenwood and Williamson (GW model for short). In order to account for the most general geometry, principles of differential geometry of surface are applied. This method requires more rigorous mathematical manipulations, as it preserves the original surface geometries (i.e. not require the original system to be replaced by any equivalent system) makes the modeling procedure much more intuitive. For subsequent use, differential geometry of axis-symmetric surface is considered instead of general surface (although this “general case” can be done as well) in Chapter 3.1. The final formulas for contact area, load, and frictional torque are derived.

One direct application of this study is the analysis of roughness-dependent frictional torque occurring rotating parts, whose geometry is often axis-symmetric. For flat surfaces contact (i.e. two flat surfaces slide across each other), effect of friction is generally quantified with the calculation of frictional force value. Similarly, for curved surface contact (e.g. in journal bearing), the value of frictional torque is frequently required. Unlike the former situation, where surface roughness does not affect the frictional force if one uses the Coulomb’s friction model (since the total reaction force at the points of asperity contact always equal to the load), surface roughness changes the distribution of contact pressure across the curved surfaces (even when Coulomb’s model holds), thus frictional torque value would be different. Furthermore, the frictional torque will not vary linearly with the load like when one models contacting surfaces smooth, rather it will also be dependent on the roughness. This topic is clarified through two specific examples. Lastly, additional analysis on the load – contact area and frictional torque – load relationship is presented.

2. MODELING ROUGH SURFACE CONTACT:
Greenwood and Williamson (1966) proposed a method to mathematically model the stochastic nature of surface’s microscopic structure by using a probabilistic approach, which introduce the concept of “asperity” and consider their height to be normally distributed over the entire rough surface. In practice, such statement is valid for most high-end engineering surfaces (i.e. homogeneous, isotropic surface), yet not quite so for other lower-end ones (Bhushan 2001). For the latter situation, Kotwal and Bhushan (1996) have developed an analytical method to generate probability density functions of non-Gaussian distributions, but will not be considered in this study. One key assumption in the work of Greenwood and Williamson is that each individual contact does not affect the deformation of its neighbors and the asperity is spherical with curvature $\mathcal{R}_0$ at its peak (Figure 1). This conveniently allows the each individual contact to be modeled independently by implementing the Hertzian theory. As mentioned previously, this method limits the contact to only be between nominally flat rough surfaces, so that the above assumption holds.
Consider two rough surfaces in contact which could be replaced by a system of two other surfaces with equivalent asperities’ curvature and RMS roughness parameter. The first surface is perfectly smooth and is located at some distance \( l_0 \) from the reference line \( h_0 \). The second surface is considered rough with the asperities’ height \( z \) varies randomly around the reference line which is described by the Gaussian distribution (Skewness = 0 and Kurtosis = 3) (Figure 2):

\[
\Phi(z) = \frac{1}{\sqrt{2\pi R_q}} \exp \left( -\frac{(z-h_0)^2}{2R_q^2} \right)
\]

(1)

where \( R_q \) is the RMS roughness parameter.

If the total number of asperities on this surface is \( N_0 \), the number of asperities having height in the interval \([z, z + dz]\) comes into contact is \( \nu = N_0 \Phi(z)dz \) and thus the total number of asperities in contact is \( N = \int_{x_1}^{x_2} \int \nu = \int_{x_1}^{x_2} \int \nu = \int_{x_1}^{x_2} \int \Phi(z)dz \).

Furthermore, according to the Hertzian contact theory, when a elastic sphere is indented to depth \( z-l_0 \) (from now refer as the indentation depth) in an elastic half-space:

The contact area is \( A_{sing} = \pi a^2 = \pi R_0 (z-l_0) \) and the required force is \( F_{sing} = \frac{4}{3} \pi R_0^2 (z-l_0)^2 \).

Thus the cumulative contact area is \( A = \int_{x_1}^{x_2} \int \nu A_{sing} = \int_{x_1}^{x_2} \int \nu = \int_{x_1}^{x_2} \int \nu = \int_{x_1}^{x_2} \int \pi R_0 (z-l_0) \times N_0 \Phi(z)dz \)

and the cumulative required force is \( F = \int_{x_1}^{x_2} \int \nu F_{sing} = \int_{x_1}^{x_2} \int \nu \times E^* R_0^2 (z-l_0)^2 \times N_0 \Phi(z)dz \) with \( E^* \) is the equivalent modulus of elasticity and can be found using \( \frac{1}{E^*} = \frac{1-E_2}{E_1} + \frac{1-E_1}{E_2} \) \( (E_1, E_2, \nu_1, \nu_2 \) are the moduli of elasticity and Poisson’s ratios of the two bodies) (Figure 3).

Noting that if a thin layer of coating is present, according to Liu et al (2005) a different equivalent modulus of elasticity should be used.
Figure 1. GW model of a single asperity

Figure 2. GW model of rough surfaces contact
Very recently, the exact solutions for these integrals have been found in Jackson and Green (2011):

If

$$I_{ep} = \int_{a}^{\infty} (z-a)^{3/2} \Phi(z) \, dz$$ with \( \Phi(z) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sigma_s} \right) \exp\left( \frac{z^2}{2\sigma_s^2} \right) \)

and

$$I_{ea} = \int_{a}^{\infty} (z-a) \Phi(z) \, dz \text{, then:}$$

$$I_{ep} = \begin{cases} \frac{1}{4\sqrt{\pi}} \left( \frac{1}{\sigma_s} \right) \sqrt{\alpha} \exp(-\gamma) \left[ (1+\alpha^2) K_{\alpha}(\gamma) - \alpha^2 K_{\alpha}(\gamma) \right] & \text{for } a > 0 \\ \frac{1}{2\sqrt{\pi}} \Gamma\left( \frac{3}{4} \right) \left( \frac{1}{\sigma_s} \right)^{1/2} & \text{for } a = 0 \\ \frac{1}{4\sqrt{2}} \left( \frac{1}{\sigma_s} \right) \frac{\sqrt{\alpha}}{\pi} \exp(-\gamma) \left[ (1+\alpha^2) I_{\alpha}(\gamma) + (3+\alpha^2) I_{\alpha}(\gamma) + \alpha^2 (I_{\alpha}(\gamma) + I_{\alpha}(\gamma)) \right] & \text{for } a < 0 \end{cases}$$ (2)

$$I_{ea} = \sqrt{\frac{1}{2\pi}} \left( \frac{1}{\sigma_s} \right) \exp(-\frac{\alpha^2}{2}) - \frac{a}{2} \text{erfc}\left( \frac{\alpha}{\sqrt{2}} \right)$$ (3)

where:

$$\alpha = \frac{a}{\sigma_s} \, , \, \gamma = \frac{\alpha^2}{4}$$

\( I(\cdot) \) and \( K(\cdot) \) are the modified Bessel function of the first & second kind respectively

\( \text{erfc}(\cdot) \) is the complimentary error function

\( \Gamma(\cdot) \) is the gamma function
3. NEARLY-IDENTICAL ROUGH CURVED SURFACES CONTACT:
Since the two curved surfaces considered in this study are nearly-identical, contact at each infinitesimal area could be treated as the contact between two nominally flat rough surface, which can then be integrated to describe the overall contact behavior between gross geometries. This approach requires the number of asperities and the indentation depth at each differential surface contact to be found. Furthermore, the magnitude, direction and location of application of contact pressure (i.e. define a bound vector) at each individual asperity as well as corresponding differential surface area should also be stated. What is known is the applied load, geometry and material properties of considered surfaces, and therefore any expression should be written in terms of these given parameters.

3.1. Differential geometry of surfaces:
i) Vector formalism of line in 3D space:
It is very convenient to express a general bound vector in 3D space using vector formulation. A bound vector is completely defined if its initial point, magnitude and direction are specified. In this problem, three quantities need to be expressed vectorially are the asperity’s direction, the reaction force and the friction force.

Consider a straight line \( L \) is defined by two parameters \((x_p, \lambda)\) (Figure 4). An arbitrary point \( Q \) on the line has position vector \( x_Q \) is given by \( x_Q = x_p + \lambda l \) where \( \lambda \) is an arbitrary scalar.

We are also interested in the point where an asperity comes into contact: a point \( I \) is the intersection of two line \( L_1 \) and \( L_2 \) has position vector \( x_I \) given by:

\[
x_I = x_1 + \lambda_1 l_1 = x_2 + \lambda_2 l_2
\]  

(4)

Figure 4. 3D line parameters
ii) Differential geometry of surface of revolution:
To understand the contact of any asperity on the surface, differential geometry is used to describe many of the surface’s characteristics as function of surface’s parameters (Figure 5).

A general surface S in 3D space can be generated by two parameters $\xi_1$ and $\xi_2$. Any point on the surface has the position vector $\mathbf{p} = \mathbf{p}(\xi_1, \xi_2)$. In orthonormal coordinates the surface of revolution can always be expressed as (Gray 1997):

$$\mathbf{p}(\xi_1, \xi_2) = \begin{bmatrix} \phi(\xi_2) \cos(\xi_1) \\ \phi(\xi_2) \sin(\xi_1) \\ \psi(\xi_2) \end{bmatrix}$$

(5)

Let $\frac{\partial \mathbf{p}_0}{\partial \xi_1} = \mathbf{a}_1$ and $\frac{\partial \mathbf{p}_0}{\partial \xi_2} = \mathbf{a}_2$:

$$\mathbf{a}_1 = \begin{bmatrix} - \phi \sin(\xi_1) & \phi \cos(\xi_1) \end{bmatrix}^T$$

$$\mathbf{a}_2 = \begin{bmatrix} - \phi' \cos(\xi_1) & \phi' \sin(\xi_1) & \psi' \end{bmatrix}^T$$

$$\nabla \mathbf{a}_2 = \begin{bmatrix} \phi(\psi' \cos(\xi_1) - \phi' \psi \sin(\xi_1)) & - \phi \phi' \end{bmatrix}^T$$

$$\left\| \nabla \mathbf{a}_2 \right\| = |\phi| \sqrt{\psi'^2 + \phi'^2}$$

The unit normal vector $\mathbf{n}$ of the surface is:
For a surface of revolution, it is natural to pick \( \xi_1 = r \), \( \xi_2 = \phi \) (Figure 6).

If a surface is obtained by rotating the curve \( z = f(r) \) from \( z = z_1 \) to \( z = z_2 \) around the \( z \)-axis, from Eq. (5) any point on this surface has the position vector:

\[
\mathbf{p}(r, \phi) = \begin{pmatrix} r \cos \phi \\ r \sin \phi \\ f(r) \end{pmatrix}
\]

with \( \phi(r) = r \) and \( \psi(r) = f(r) = z \).

From Eq. (6) and Eq. (7), the corresponding unit normal vector \( \mathbf{n} \) and the corresponding area of a differential surface element \( dA \) is:

\[
\mathbf{n}(\xi_1, \xi_2) = \mathbf{\varepsilon}_1 \mathbf{a}_2 \frac{\text{sgn}(\phi)}{\sqrt{\phi'^2 + \psi'^2}} \begin{pmatrix} \psi' \cos(\xi_1) \\ \psi' \sin(\xi_1) \end{pmatrix}
\]

\[
dA = \left| \mathbf{\varepsilon}_1 \mathbf{a}_2 \right| d\xi_1 d\xi_2 = \text{sgn}(\phi) \sqrt{\phi'^2 + \psi'^2} d\xi_1 d\xi_2
\]
\[ dA = \sqrt{1 + (z')^2} \, rdrd\varphi \]  \hspace{1cm} (10)

Furthermore, if \( g(z) = f^{-1} \), the (apparent) area of this surface given by Anton (1999) is:

\[ A_{\text{upper}} = 2\pi \int_{z_1}^{z_2} g(z) \sqrt{1 + [g'(z)]^2} \, dz \]  \hspace{1cm} (11)

3.2. Calculation of contact area, contact pressure and frictional torque:

Consider a surface of revolution S1 with vertex \( O_1 \) is at the origin of frame \( \mathcal{R}_1 = (\vec{i}_1, \vec{j}_1, \vec{k}_1) \) and a surface of revolution S2 with vertex \( O_2 \) is at the origin of frame \( \mathcal{R}_2 = (\vec{i}_2, \vec{j}_2, \vec{k}_2) \) (Figure 7). \( (\vec{i}_1, \vec{i}_2, \vec{i}_3) \) can always be chosen such that the position vector of \( O_2 \) with respect to \( \mathcal{R}_1 \) is the eccentricity vector \( \vec{e} = [e_x \quad 0 \quad e_z]^T \). Because the surface is axis-symmetric and the asperities are randomly distributed, the eccentricity vector is assumed to be parallel to the load \( \vec{F} = [F_r \quad 0 \quad F_z]^T \), which is a known vector.

![Figure 7. Contacting surfaces of revolution](image)

According to the GW model, let S1 is a rough surface and S2 is a perfectly smooth surface. The geometries of S1 and S2 are generated by rotating the curves \( z_1 = f_1(r) \) and \( z_2 = f_2(r) \) around \( \vec{i}_3 \) and \( \vec{j}_3 \), respectively.
Consider a differential area at point $I_1$ on the rough surface $S_1$ defined by two parameters $(r_1, \phi_1)$ and associated with the normal vector $\overline{n_1}(r_1, \phi_1)$ (resolved in $\Re_1$). $\overline{n_1}$ intersects the smooth surface $S_2$ at point $I_2$. $\overline{n_2}(r_2, \phi_2)$ is normal vector of the smooth surface $S_2$ at point $I_2$ (resolved in $\Re_2$). From Eq. (9) and Eq. (10):

$$
\overline{n_1} = \left[ \frac{z_1' \cos(\phi_1)}{\sqrt{1 + (z_1')^2}} \quad \frac{z_1' \sin(\phi_1)}{\sqrt{1 + (z_1')^2}} \quad \frac{1}{\sqrt{1 + (z_1')^2}} \right]^T = \left[ n_{1,x} \quad n_{1,y} \quad n_{1,z} \right]^T
$$

(12)

$$
\overline{n_2} = \left[ \frac{z_2' \cos(\phi_2)}{\sqrt{1 + (z_2')^2}} \quad \frac{z_2' \sin(\phi_2)}{\sqrt{1 + (z_2')^2}} \quad \frac{1}{\sqrt{1 + (z_2')^2}} \right]^T = \left[ n_{2,x} \quad n_{2,y} \quad n_{2,z} \right]^T
$$

(13)

$$
dA = \sqrt{1 + (z_1')^2} r_1 dr_1 d\phi_1
$$

(14)

Introduce $\Delta(r_1, \phi_1) = d(I_1, I_2)$ and $\delta = d(I_1, P)$ with $P$ is any point on the line $I_1I_2$. Physically, $\delta$, $\overline{n_1}$, $\overline{n_2}$ represent the height, the direction of an individual asperity and the direction of reaction force respectively (Figure 8).
System of equations for the intersection at \( I_2 \) can then be derived from Eq. (4) and Eq. (8):

\[
\begin{align*}
\begin{bmatrix}
e_x + r_2 \cos(\phi_2) \\
r_2 \sin(\phi_2) \\
e_z + z_2
\end{bmatrix}
&= \begin{bmatrix}
\Delta(r_1, \phi_1) n_{i,x} + r_1 \cos(\phi_1) \\
\Delta(r_1, \phi_1) n_{i,y} + r_1 \sin(\phi_1) \\
\Delta(r_1, \phi_1) n_{i,z} + z_1
\end{bmatrix} \\
&= \begin{bmatrix}
\Delta(r_1, \phi_1) n_{i,x} + r_1 \cos(\phi_1) \\
\Delta(r_1, \phi_1) n_{i,y} + r_1 \sin(\phi_1) \\
\Delta(r_1, \phi_1) n_{i,z} + z_1
\end{bmatrix}
\end{align*}
\]  

(15)

By solving this system, we obtain \( \Delta(r_1, \phi_1) \) as well as \( r_1 \), \( r_2 \). Substitute \( r_1 \), \( r_2 \) into Eq. (13), we could express \( n_2 \) in terms of \( r_1 \) and \( \phi_1 \). The indentation depth of an individual asperity can be found as:

\[
\mathcal{Z}(r_1, \phi_1) = \mathcal{R}_0 - (\Delta + \mathcal{R}_0 - \delta) \cos(\theta)
\]

(16)

with \( \cos(\theta) = \vec{n}_1 \cdot \vec{n}_2 \) (Figure 9).

In a differential area, the asperities can be assumed to be unidirectional (i.e. having the same direction vector). If the asperities density is \( \mathcal{N}_0 \), The number of asperities in that differential area is \( \mathcal{N}_0 dA = \mathcal{N}_0 \sqrt{1 + (z_1')^2} r_1 dr_1 d\phi_1 \). Since the asperities’ height is normally distributed described by the Gaussian distribution, the number of asperities on a differential area that height in the interval \([\delta, \delta + d\delta]\) is:

\[
N = \mathcal{N}_0 dA \times \Phi(\delta) d\delta = \mathcal{N}_0 \Phi(\delta) \sqrt{1 + (z_1')^2} r_1 dr_1 d\phi_1 d\delta
\]

(17)

In terms of contact pressure distribution:

\[
\Pi = \frac{N \times F}{dA} = \frac{4}{3} E' \mathcal{R}_0^{1/2} \mathcal{N}_0 \Phi(\delta) \mathcal{Z}^{1/2} d\delta \times \vec{n}_2
\]

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\]

Furthermore, the position vector of a single point of contact \( I_1 \) and the moment arm (Figure 10) respectively are:
\[ p_{\sin g} = p_1 + \Delta(r_1, \varphi_1) n_1 \]

\[ p'_{\sin g} = p_{\sin g} - (p_{\sin g}^T i_3) i_3 \]

Assume the relative angular velocity is \( \varpi \), the directional unit vector of the frictional force at an asperity contact is:

\[
\begin{bmatrix}
\text{sgn}(\varpi) \text{sgn}(n_{2,x}) n_{2,y} \\
\text{sgn}(\varpi) \text{sgn}(n_{2,y}) n_{2,x} \\
0
\end{bmatrix}
\begin{bmatrix}
\gamma \quad \gamma' \quad 0
\end{bmatrix}
\]

\[ (n_x \ n_y \ n_z = 0) \]  

Finally, we attain the expressions for the number of asperities, real contact area, reaction force components and frictional torque in terms of an individual asperity, a differential surface area and the entire contacting surfaces:

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Figure 9. Contact of a single asperity
Table 1. Expressions for the number of asperities, real contact area, reaction force components and frictional torque

<table>
<thead>
<tr>
<th></th>
<th>Single asperity</th>
<th>Differential surface area</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number of asperities in contact</strong></td>
<td>$V = N_0 \Phi(\delta) \sqrt{1 + (z')^2} r_i dr_i d\phi_i d\delta$</td>
<td>$2\int_0^\infty \int_0^\infty V = N_c$</td>
</tr>
<tr>
<td><strong>Real contact area</strong></td>
<td>$A_{\text{sin g}} = \pi R_0 h$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A_{\text{diff}} = V \times A_{\text{sin g}} = \pi R_0 N_0 \Phi(\delta) h \sqrt{1 + (z')^2} r_i dr_i d\phi_i d\delta$</td>
<td>$2\int_0^\infty \int_0^\infty \int_0^\infty A_{\text{diff}} = A_{\text{real}}$</td>
</tr>
<tr>
<td><strong>Reaction force</strong></td>
<td>$F_{\text{sin g},x} = \frac{4}{3} \pi R_0^3 \frac{h}{2} \times (n_2^T \vec{t}_1)$</td>
<td>$2\int_0^\infty \int_0^\infty \int_0^\infty F_{\text{diff},x} = F_x$</td>
</tr>
<tr>
<td>(x - component)</td>
<td>$F_{\text{diff},x} = \frac{4}{3} \pi R_0^3 \frac{h}{2} \times (n_2^T \vec{t}_1)$</td>
<td>$2\int_0^\infty \int_0^\infty \int_0^\infty F_{\text{diff},x} = 0$</td>
</tr>
<tr>
<td><strong>Reaction force</strong></td>
<td>$F_{\text{sin g},y} = \frac{4}{3} \pi R_0^3 \frac{h}{2} \times (n_2^T \vec{t}_2)$</td>
<td>$2\int_0^\infty \int_0^\infty \int_0^\infty F_{\text{diff},y} = F_y$</td>
</tr>
<tr>
<td>(y - component)</td>
<td>$F_{\text{diff},y} = \frac{4}{3} \pi R_0^3 \frac{h}{2} \times (n_2^T \vec{t}_2)$</td>
<td>$2\int_0^\infty \int_0^\infty \int_0^\infty F_{\text{diff},y} = 0$</td>
</tr>
<tr>
<td><strong>Reaction force</strong></td>
<td>$F_{\text{sin g},z} = \frac{4}{3} \pi R_0^3 \frac{h}{2} \times (n_2^T \vec{t}_3)$</td>
<td>$2\int_0^\infty \int_0^\infty \int_0^\infty F_{\text{diff},z} = F_z$</td>
</tr>
<tr>
<td>(z - component)</td>
<td>$F_{\text{diff},z} = \frac{4}{3} \pi R_0^3 \frac{h}{2} \times (n_2^T \vec{t}_3)$</td>
<td>$2\int_0^\infty \int_0^\infty \int_0^\infty F_{\text{diff},z} = 0$</td>
</tr>
<tr>
<td><strong>Frictional torque</strong></td>
<td>$T_{\text{sin g}} = \frac{4}{3} \mu E^* \pi R_0^3 \frac{h}{2} \times (\vec{p}_{\text{sin g}} \vec{n}_2)$</td>
<td>$2\int_0^\infty \int_0^\infty \int_0^\infty T_{\text{diff}} = T_f$</td>
</tr>
<tr>
<td></td>
<td>$T_{\text{diff}} = \nu \times T_{\text{sin g}} = \frac{4}{3} \mu E^* \pi R_0^3 \frac{h}{2} \times (\vec{p}_{\text{diff}} \vec{n}_2)$</td>
<td></td>
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</table>
Eq. (20) shows the number $N_c$ of asperities that are in contact. Eq. (21) can be solved using Eq. (3) that approximates the real contact area. Eq. (22), (23), (24) yield the components of the cumulative reaction force which according to Newton’s third law have to be equal to the components of the applied load. Finally, Eq. (25) gives the expression for the cumulative frictional torque.

![Figure 10. Representation of moment arm](image)

**4. APPLICATION TO SOME COMMON SURFACE GEOMETRIES:**

4.1. Concentric spherical annulus:

Consider two concentric spherical surfaces (Figure 12) created by the revolution of:

$$z_1 = f_1(r_1) = \sqrt{R_1^2 - r_1^2} \Rightarrow z_1' = \frac{-r_1}{\sqrt{R_1^2 - r_1^2}}$$  \hspace{1cm} (26)

$$z_2 = f_2(r_2) = \sqrt{R_2^2 - r_2^2} \Rightarrow z_2' = \frac{-r_2}{\sqrt{R_2^2 - r_2^2}}$$  \hspace{1cm} (27)

with:

$$F = \begin{bmatrix} 0 & 0 & F^{sph} \end{bmatrix}$$  \hspace{1cm} (28)

$$\varepsilon = \begin{bmatrix} 0 & 0 & \varepsilon \end{bmatrix} (\varepsilon < 0)$$  \hspace{1cm} (29)
\[ R_2 \approx R_1 - \varepsilon \left( |\varepsilon| \ll R_1, R_2 \right) \quad (30) \]

Figure 11. Schematic of a concentric spherical annulus

Dropping the subscript of \( r_1, \phi_1 \). It can be readily verified that \( \Delta(r, \phi) = -\varepsilon \) and:

\[
\bar{n}_1 = \bar{n}_2 = \begin{bmatrix}
-\frac{r}{R_1} \cos(\phi) & -\frac{r}{R_1} \sin(\phi) & \frac{\sqrt{R_2^2 - r^2}}{R_1}
\end{bmatrix}^T
\]

Thus from Eq. (16) and Eq. (18):

\[ \mathfrak{F}(r, \phi) = \delta + \varepsilon \quad (32) \]

\[ \frac{p'_{\text{sing}}}{\sin \phi} = \begin{bmatrix}
\left( 1 - \frac{\varepsilon}{R_1} \right) r \cos(\phi) & \left( 1 - \frac{\varepsilon}{R_1} \right) r \sin(\phi) & 0
\end{bmatrix}^T
\]

\[ \left\| \frac{p'_{\text{sing}}}{\sin \phi} \right\| \approx \left\| p'_{\text{sing}} \right\| = \left( 1 - \frac{\varepsilon}{R_1} \right) r \quad (34) \]

Substitute Eq. (1), (26), (32), (34) into Eq. (25), we attain the frictional torque expression:
\[ \| I_{f}^{sph} \| = \frac{4}{3} \mu E^{*} N_{0}^{1/2} N_{0} R_{2} \times I_{f} \]  

(35)

where:

\[ I_{\phi} = \int_{0}^{2\pi} d\phi = 2\pi \]

\[ I_{\delta} = \int_{\Delta=\varepsilon}^{\infty} (\delta + \varepsilon)^{3/2} \frac{1}{\sqrt{2\pi R_{q}}} \exp \left( -\frac{\delta^{2}}{2R_{q}^{2}} \right) d\delta \]

\[ I_{r} = \int_{d_{1}}^{d_{2}} \frac{r^{2}}{\sqrt{R_{1}^{2} - r^{2}}} dr = \frac{1}{2} \left( R_{1}^{2} \arctan \left( \frac{r}{\sqrt{R_{1}^{2} - r^{2}}} \right) - r \sqrt{R_{1}^{2} - r^{2}} \right) \]  

(36)  

(37)

4.2. Eccentric cylindrical annulus:
Consider two cylindrical surfaces (Figure 13) defined by the revolution:

\[ r_{1} = R_{1} \Rightarrow z_{1}' = \infty \]  

\[ r_{2} = R_{2} \Rightarrow z_{2}' = \infty \]  

with:

\[ F = \begin{bmatrix} F^{cyl} \cos(\phi_{1}) & F^{cyl} \sin(\phi_{1}) & 0 \end{bmatrix} \]  

(38)

\[ \varepsilon = \begin{bmatrix} \varepsilon \cos(\phi_{1}) & \varepsilon \sin(\phi_{1}) & 0 \end{bmatrix} \]  

(39)

\[ dA = hR_{d}d\phi_{1} \]  

(40)

Eq. (15) then becomes:

\[ \begin{bmatrix} \varepsilon + R_{2} \cos(\phi_{2}) \\ R_{2} \sin(\phi_{2}) \\ h_{2} \end{bmatrix} = \begin{bmatrix} \Delta(R_{1},\phi_{1}) \cos(\phi_{1}) + R_{1} \cos(\phi_{1}) \\ \Delta(R_{1},\phi_{1}) \sin(\phi_{1}) + R_{1} \sin(\phi_{1}) \\ h_{1} \end{bmatrix} \]

\[ \begin{bmatrix} \Delta(R_{1},\phi_{1}) \cos(\phi_{1}) + R_{1} \cos(\phi_{1}) \\ \Delta(R_{1},\phi_{1}) \sin(\phi_{1}) + R_{1} \sin(\phi_{1}) \\ h_{1} \end{bmatrix} \]  

(41)  

(42)  

(43)
Squaring \((a)\) and \((b)\) then take their sum:

\[
R_2^2 = (\Delta \cos(\varphi_1) + R_1 \cos \varphi_1 - \varepsilon)^2 + (\Delta \sin(\varphi_1) + R_1 \sin \varphi_1)^2
\]

\[
\iff R_2^2 = \Delta^2 + 2\Delta R_1 - 2\Delta \varepsilon \cos(\varphi_1) + R_1^2 - 2\varepsilon R_1 \cos(\varphi_1) + \varepsilon^2
\]

\[
\iff \Delta^2 + [2R_1 - 2\varepsilon \cos(\varphi_1)]\Delta + [R_1^2 - R_2^2 - 2\varepsilon R_1 \cos(\varphi_1) + \varepsilon^2
\]

After solving for \(\Delta\) and dropping higher order terms of \(\varepsilon\), we get:

\[
\Delta \approx R_2 - R_1 + \varepsilon \cos(\varphi_1) \tag{41}
\]

Solve for \(\cos(\varphi_2), \sin(\varphi_2)\) then substitute into Eq. (12) and Eq. (13):

\[
\bar{n}_1 = \begin{bmatrix} \cos(\varphi_1) & \sin(\varphi_1) & 0 \end{bmatrix}^T
\]

\[
\bar{n}_2 = \begin{bmatrix} \cos(\varphi_1) - \frac{\varepsilon}{R_2} \sin^2(\varphi_1) & \sin(\varphi_1) + \frac{\varepsilon}{R_2} \cos(\varphi_1) \sin(\varphi_1) \end{bmatrix}
\]

\[
\Rightarrow \bar{n}_1^T \bar{n}_2 = 1
\]

Thus from Eq. (16), (18) and (19):

\[
\Im(r_1, \varphi_1) = \delta - \Delta = \delta - (R_2 - R_1 + \varepsilon \cos(\varphi_1)) \tag{44}
\]

\[
\bar{p}'_{\text{sing}} = \begin{bmatrix} (R_1 + \Delta) \cos(\varphi_1) & (R_1 + \Delta) \sin(\varphi_1) & 0 \end{bmatrix}^T
\]

\[
\left\|\bar{p}'_{\text{sing}} n_f\right\| \approx \left(2R_2^2 \sin^2(\varphi_1) \cos^2(\varphi_1) + 2\varepsilon \sin^2(\varphi_1) \cos^3(\varphi_1) \right) \tag{46}
\]

Substitute Eq. (1), (35), (44), (46) into Eq. (21), (22), (25), we attain the contact area, applied load and cumulative frictional torque expressions:

\[
A'^{\psi l} = \pi R_0 \Sigma_0 \frac{1}{\sqrt{2\pi} R_q} \int_0^{2\pi} R_1 h \int_0^\delta \exp \left(-\frac{\delta^2}{2R_q^2}\right) (\delta - (R_2 - R_1 + \varepsilon \cos(\varphi_1)) \cos(\varphi_1) + \varepsilon \sin(\varphi_1)) \tag{47}
\]
\[
F_{cyl} = \frac{4}{3} E^* R^\frac{3}{2} N_0 \frac{1}{\sqrt{2\pi R_q}} R_i h \times \left[ \int_{0}^{2\pi} \exp \left( -\frac{\delta^2}{2R_q^2} \right) \left( \delta - \left( R_2 - R_1 + \varepsilon \cos(\phi_i) \right) \right)^{3/2} \cos(\theta) \right]
\]

\[
\left\| T_{cyl} \right\| = \frac{4}{3} \mu E^* R^\frac{3}{2} N_0 \frac{1}{\sqrt{2\pi R_q}} R_i h \times \left[ \int_{0}^{2\pi} \exp \left( -\frac{\delta^2}{2R_q^2} \right) \left( \delta - \left( R_2 - R_1 + \varepsilon \cos(\phi_i) \right) \right)^{3/2} \right. \\
\left. \times \left( 2R_2^2 \sin^2(\phi_i) \cos^3(\phi_i) + 2\varepsilon \left( 3\sin^2(\phi_i) \cos^3(\phi_i) \right) - s \right) \right]
\]

5. ANALYSIS:

However, the eccentricity (\( \varepsilon \)) is hard to measure without special instruments. Thus it is more convenient to consider the frictional torque – load relationship. By eliminating \( \varepsilon \) using Eq. (22) to Eq. (25), we can find:

\[
\left\| T_{f} \right\| = \tau(F, \mu, R_q, G_i)
\]

where:

- \( F \) and \( \mu \) are the applied load and the friction coefficient
- \( R_q \) is RMS roughness parameter
- \( G_i \)'s are the surfaces geometric parameters

\[
\left\| T_{f}^* \right\| = \tau'(G_i) \mu \left\| F \right\|
\]

Which means the frictional torque is linearly dependent on the applied load value. It is expected that \( \left\| T_{f}^* \right\| \approx \left\| T_{f} \right\| \) and the two values become closer if more assumptions about the geometries are made. For example, in Section IV.1, if we assume \( R_2 \approx R_1 - \varepsilon \) (\( |\varepsilon| \ll R_1, R_2 \)), then from Eq.(24) and Eq. (25):
\[ A^{\text{cy}l} = \pi R_0 \sum_{n_0} \frac{1}{\sqrt{2\pi R_q}} R_l h \times \int_{0}^{2\pi} \exp \left( -\frac{\delta^2}{2R_q^2} \right) (\delta - (R_2 - R_1) + \delta) \] (52)

After replacing \( d_1 = R_1 \sin(\theta_1) \), \( d_2 = R_1 \sin(\theta_2) \), Eq. (50) becomes:

\[ A^{\text{cy}l} = \pi R_0 \sum_{n_0} \frac{1}{\sqrt{2\pi R_q}} R_l h \times \int_{0}^{2\pi} \exp \left( -\frac{\delta^2}{2R_q^2} \right) (\delta - (R_2 - R_1) + \delta) \] (53)

Eq. (53) is exactly the formula for the frictional torque – applied load relationship if the contacting surfaces are modeled as smooth that is obtained in several studies such as the one from Grégory (2014).

Considering the example in Section 4.2 where no additional geometric assumption is made. First, the consistency between theories of contact mechanics should be taken into account. In the early days, Hertz set the foundation of contact mechanics by analytically predicting the compressive force required to indent a smooth sphere into an infinite smooth half-space, which was then broadened to account for other shapes. According to Sneddon 1965, in the case of parallel-axis cylinders contact, the applied force as function of indentation depth \( \varepsilon + R_l - R_q \) is:

\[ F_{\text{smooth}} = \pi h (\varepsilon + R_l - R_q) / 4 \] (54)

Thus Eq. (48) should be identical to Eq. (54) as \( R_q \rightarrow 0 \) since both describe contact in the “smooth” case. Taking the limit of Eq. (48):

\[ F_{\text{smooth}}(0) = \lim_{R_q \rightarrow 0} F^{\text{cy}l} = \frac{4}{3} F^{\text{cy}l} R_0 N_p \int \int_{R_q} \left( \int \frac{1}{\sqrt{2\pi R_q}} \exp \left( -\frac{\delta^2}{2R_q^2} \right) (\delta - (R_2 - R_1 + \varepsilon \cos(\phi)) + \frac{\varepsilon}{R_q} \sin^{1/2}(\phi)) d\delta d\phi \right) \] (55)

Noting that the Gaussian distribution function becomes the Dirac delta function as \( R_q \rightarrow 0 \), which has a special property:

\[ \int \delta_\alpha f(x) = f(0) \] (55)
Using this property, Eq. (54) becomes:

\[ F_{\text{rough}}(0) = \frac{4}{3} E^* R_o^{\frac{3}{2}} \left[ R_0 R_1 h \times \int_0^{2\pi} \left( R_1 - R_2 - \varepsilon \cos(\varphi_i) \right)^{3/2} \left( \cos(\varphi_i) - \frac{\varepsilon}{R_2} \sin^2(\varphi_i) \right) d\varphi \right] \]

\[ I_{\varphi_i} = \int_0^{2\pi} \left( R_1 - R_2 - \varepsilon \cos(\varphi_i) \right)^{3/2} \left( \cos(\varphi_i) - \frac{\varepsilon}{R_2} \sin^2(\varphi_i) \right) d\varphi \]

Let

The analytical solution for this definite integral could be found using Wolfram Alpha:

\[ I_{\varphi_i} = \frac{2\sqrt{a + b}}{105b^2} \left\{ (6a^2c - 21a^2b + 58ab^2c - 63b^3) \left[ E \left( 2\pi \left( \frac{2b}{a + b} \right) \right) - E \left( 0 \left( \frac{2b}{a + b} \right) \right) \right] \right. \\
\left. - (a - b)(6a^2c - 21ab + 10b^2c) \left[ F \left( 2\pi \left( \frac{2b}{a + b} \right) \right) - F \left( 0 \left( \frac{2b}{a + b} \right) \right) \right] \right\} \quad (56) \]

where:

\( a = R_2 - R_1, \quad b = \varepsilon, \quad c = \varepsilon / R_2 \)

\( F(\cdot) \) and \( E(\cdot) \) are the incomplete elliptic integrals of first and second kind

Since it is required that \( F_{\text{rough}}(0) = F_{\text{smooth}}, \) we attain:

\[ \frac{4}{3} E^* R_o^{\frac{3}{2}} R_0 h = \frac{\pi h(e + R_1 - R_2)}{4I} \quad (57) \]

Next, using Eq. (2), (3) we could analytically evaluate the first integrations in Eq. (47), (48), (49). Based on these expressions, a MATLAB script is written to numerically evaluate Eq. (47), (48) and (49) at increments of \( e \) and \( R_q \). The integration command “integral” uses global adaptive quadrature method “integral” with absolute error tolerance of 1e-10 (Shampine, 2008).

Graphs 1 to Graph 6 show the relationships between contact area, applied force, frictional torque with eccentricity and roughness. Graph 7 indicates that even though the contact area – applied load relationship might be linear at a particular roughness, the contact area increases faster with applied load as roughness goes up. We even observe this behavior more clearly in the case of frictional torque – applied load relationship.
Graph 1. Contact area as function of eccentricity and roughness

Graph 2. Contact area as function of eccentricity at various roughness
Graph 3. Applied force as function of eccentricity and roughness

Graph 4. Applied force as function of eccentricity at various roughness
Graph 5. Frictional torque as function of eccentricity and roughness

Graph 6. Frictional torque as function of eccentricity at various roughness
Graph 7. Contact area as function of applied force at various roughness

Graph 8. Frictional torque as function of applied force at various roughness
6. CONCLUSION

The paper proposes a method to account for the contact of rough curved surface having nearly identical geometries. General formulas for the true contact area as well as the applied force have been deduced in terms of definite integrals. These integrals could be simplified analytically. Numerical technique and programming are then implemented in order to perform analysis of a special case: two cylindrical rough surfaces in contact.

REFERENCES
